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Subinvariant Integrals and Means on Topological Semigroups

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1. INTRODUCTION

Let S be a locally compact Hausdorff semigroup with the property (#): for each compact set $K \subset S$ and each element $a \in S$, the set

$$Ka^{-1} = \{x \in S : xa \in K\}$$

is compact.

Let $K(S)$ denote the space of all continuous real-valued functions on S with compact supports. For $f \in K(S)$ and $a \in S$, define $f_a(x) = f(xa)$ for all $x \in S$. The condition (#) on S implies that $f_a \in K(S)$ for all $f \in K(S)$ and $a \in S$. A positive linear functional I on $K(S)$ is called a right subinvariant integral if

$$I(f_a) \leqslant I(f) \tag{1}$$

for each nonnegative function $f \in K(S)$ and $a \in S$. The concept of subinvariant integral is very much similar to the concept of semiinvariant exterior density relative to a transformation on the space as defined in [2, Section 8] and also to the concept of a semiinvariant measure as defined in [5].

I is said to be right invariant if the equality holds in (1). The question of existence of right invariant integrals on S has been investigated by Michael [3] and Argabright [1]. In this paper we discuss conditions under which the subinvariance of I implies its invariance. In the proof [1, Theorem 1] it is shown that the support of every right invariant integral on S is contained in every closed left ideal of S. However the support of a right subinvariant integral need not be contained in all closed left ideals of S. For example, consider $S = [1, \infty)$ with the usual topology and the usual multiplication. Let I be the Lebesgue integral on S. We observe that I is subinvarinat, for if $f \in K(S)$, $f \geqslant 0$ and $a \in S$, we have

$$\begin{aligned} I(f_a) &= \int_1^\infty f(ax)\,dx \\ &= \frac{1}{a}\int_a^\infty f(x)\,dx \leqslant \int_1^\infty f(x)\,dx \\ &= I(f). \end{aligned}$$

However, the support of I is equal to S which is not contained in any proper closed left ideal of S. This also implies that I is not right invariant.

In what follows we shall prove that the essential difference between the right invariant and the right subinvariant integrals lies in the fact that the support of the former is contained in all the closed left ideals of S whereas the support of the later may not be contained in all the closed left ideals of S. As a consequence we obtain topological and algebraic conditions on S so that every right subinvariant integral on S is right invariant. We also obtain a measure theoretic condition on a right subinvariant integral I on S so that it is right invariant.

Now, let S be a topological semigroup, and let $C(S)$ denote the space of all bounded continuous real-valued functions on S. A mean on S is a positive linear functional M on $C(S)$ such that $M(f) = 1$ if $f(x) = 1$ for all $x \in S$. The right subinvariance and the right invariance of M is defined by the relations

$$M(f_a) \leqslant M(f) \qquad \text{and} \qquad M(f_a) = M(f)$$

for all nonnegative functions f in $C(S)$ and for all $a \in S$, respectively. Using the results of Markoff [2] we shall prove that every right subinvariant mean on a normal topological semigroup S is right invariant.

2. Right Subinvariant Integrals on Locally Compact Semigroups

Throughout this section we denote by S a locally compact semigroup with the property (#), by I a right subinvariant integral on S and by μ the regular Borel measure on S corresponding to the integral I.

Proposition 2.1. *I is right invariant if and only if the support of I is contained in a closed right ideal of S which is a left group.*

Proof. In the proof [1, Theorem 1] it is shown that the support of every right invariant integral on S is contained in a closed right ideal of S which is a left group. This implies the necessity of the condition in our proposition. To prove the sufficiency of the hypothesis, let us suppose that the support of I is contained in a closed right ideal D which is also a left group. It is well known that D has idempotents and if e is an idempotent of D, then e is a right identity of D and eD is a group. We shall show that for $f \in K(S)$, $f \geqslant 0$ and $a \in S$, $I(f_a) = I(f)$. Fix an idempotent e in D. Since $f = f_e$ on D, we have $I(f) = I(f_e)$. Also since the idempotent $e \in D$ which is a right ideal, we have $ea = e \cdot ea \in eD$. Since eD is a group there exists $a' \in eD$ such that $a'(ea) = (ea)\, a' = e$. Now,

$$I(f) = I(f_e) = I(f_{a'ea}) \leqslant I(f_{ea}) \leqslant I(f_a) \leqslant I(f).$$

Therefore, $I(f) = I(f_a)$. Since it is verified that $I(f) = I(f_a)$ for all $f \in K(S)$, $f \geqslant 0$ and for all $a \in S$, we conclude that I is right invariant.

THEOREM 2.2. *A right subinvariant integral I on S is right invariant if and only if its support is contained in every closed left ideal of S.*

Proof. In the proof [1, Theorem 1] it is shown that the support of every right invariant integral is contained in every closed left ideal of S. This implies the "only if" part of the Theorem.

For the "if" part of the theorem, we observe that if

$$L_0 = \bigcap \{L : L \text{ is a closed left ideal of } S\},$$

then the support of $I \subset L_0$ and since $I \neq 0$, we have $L_0 \neq \varnothing$. Now, as in the proof [1, Theorem 1], it can be proved that L_0 is a closed right ideal which is a left group. The Theorem now follows from the Proposition 2.1.

LEMMA 2.3. $\mu(K) \leqslant \mu(Ka)$ *for every compact subset K of S and* $a \in S$.

Proof. Let K be a compact subset of S and let $\epsilon > 0$ be given. By regularity of μ there exists an open set U containing K such that $\mu(U) < \mu(K) + \epsilon$. Since S is locally compact there exists a function $f : S \to [0, 1]$ such that f is 1 on K and vanishes outside U. Now,

$$\mu(Ka^{-1}) \leqslant \int_S f(ta)\, d\mu(t) = I(f_a) \leqslant I(f) \leqslant \mu(U) < \mu(K) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\mu(Ka^{-1}) \leqslant \mu(K) \qquad \text{for all } a \in S.$$

Since K is compact and $K \subset (Ka)\, a^{-1}$, it follows that Ka is compact, and

$$\mu(K) \leqslant \mu((Ka)\, a^{-1}) \leqslant \mu(Ka)$$

for every a in S.

LEMMA 2.4. *If* $\mu(S) < \infty$, *then I is right invariant.*

Proof. Let F be the support of μ. In view of the Lemma 2.3 and [5, Theorem 1], we can prove that $\overline{Fx} = F$ for every $x \in S$. Let L be a closed left ideal of S and $a \in L$. Since $F = \overline{Fa} \subset L$, therefore F is contained in every closed left ideal of S and the lemma follows from the Theorem 2.2.

COROLLARY 2.5. *Let I be a right subinvariant integral on S, and let* μ *be the corresponding regular Borel measure on S. Then I is right invariant in case* (i) $\mu(S) < \infty$, *or* (ii) *S is compact, or* (iii) *S is left simple.*

Proof. (i) and (ii) follow from the Lemma 2.4 and (iii) is an immediate consequence of the Theorem 2.2.

3. Right Subinvariant Means on Topological Semigroups

Let S be a normal topological semigroup, $C(S)$ be the space of all bounded continuous real valued functions on S and M be a mean on S.

It is shown in [2, Section 5] that M determines a unique set function μ defined on the class of all subsets of S such that

(i) $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ for $A, B \subset S$,

(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$, if $\bar{A} \cap B = \phi$,

(iii) $\mu(S) = 1$

and

(iv) $\mu(A) = \inf\{\mu(G) : G \text{ open and } G \supset A\}$.

A set function μ with the above properties is called an exterior density on S. An exterior density μ on S is said to be right semi-invariant if $\mu(Ua^{-1}) \geqslant \mu(U)$ for every open set U in S and a in S.

The following is a consequence of [2, Theorem 26]:

Theorem 3.1. *Let M be a mean on S, and let μ be the exterior density determined by M. If μ is right semi-invariant, then M is right invariant.*

Lemma 3.2. *Let M be a right sub-invariant mean on S, and let μ be the corresponding exterior density. Then μ is right semi-invariant.*

Proof. Firstly, we show that $\mu(Ka^{-1}) \leqslant \mu(K)$ for every closed subset K of S and every a in S. To show this, for arbitrary $\epsilon > 0$, choose an open set U containing K such that $\mu(U) < \mu(K) + \epsilon$.

Since S is normal there exists an open set V such that $K \subset V \subset \bar{V} \subset U$. Now, choose f in $C(S)$ such that f maps S into $[0, 1]$, is identically 1 on $\bar{V}$ and vanishes outside U. Now $Ka^{-1} \subset Va^{-1}$, and Va^{-1} is open. Also $f_a(x) = f(xa) = 1$ for every $x \in Va^{-1}$. Therefore for every $g \in C(S)$ which vanishes outside Va^{-1} and is such that $0 \leqslant g(x) \leqslant 1$ for every $x \in S$, we have $g \leqslant f_a$. Hence

$$\mu(Ka^{-1}) \leqslant \mu(Va^{-1}) \leqslant M(f_a) \leqslant M(f) \leqslant \mu(U) < \mu(K) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary it follows that $\mu(Ka^{-1}) \leqslant \mu(K)$.

Next, let U be an open set in S and $a \in S$. Then

$$\begin{aligned}\mu(Ua^{-1}) &= 1 - \mu(S\backslash Ua^{-1}) \\ &= 1 - \mu((S\backslash U)\, a^{-1}) \\ &\geqslant 1 - \mu(S\backslash U) \\ &= \mu(U).\end{aligned}$$

Therefore,

$$\mu(Ua^{-1}) \geqslant \mu(U).$$

This completes the proof of the lemma.

Theorem 3.1 and the Lemma 3.2 together imply the following:

THEOREM 3.3. *A right subinvariant mean on a normal topological semigroup is right invariant.*

Remark. Theorem 3.3 generalizes [4, Theorem 4.1.6].

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